Technical Report No. 32-57

Flow of an Electrically Conducting Viscous Fluid Past a Current-Carrying Finite Flat Plate

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CONTENTS

ABSTRACT

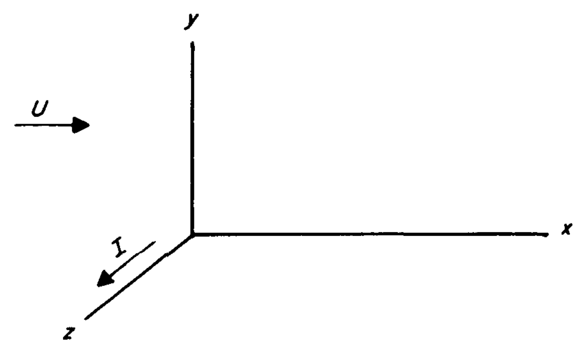
A theory is developed to describe the magnetohydrodynamic flow past a current-carrying flat plate. It is found that the magnetohydrodynamic contribution to the drag is proportional to the square of the current. A method for computing the flow field is presented. Such a flow field for liquid sodium is now being computed by the computing section at JPL.

I. INTRODUCTION

This study is in two parts. Sections II to V contain a somewhat abbreviated discussion of a problem studied elsewhere by the author (Ref. 1), namely, the two-dimensional steady flow of a viscous incompressible fluid possessing low electrical conductivity past a current element. Sections VI to IX discuss the application of such a flow past a finite, current-carrying flat plate disposed parallel to the free stream flow.

II. STATEMENT OF THE CURRENT ELEMENT PROBLEM

Consider the two-dimensional steady flow of an incompressible fluid possessing kinematic viscosity ν and electrical conductivity σ. At the origin of coordinates (see Sketch 1), there is a current element carrying I amperes in a direction normal to the flow (toward the viewer in Sketch 1). The flow velocity $\mathbf{q}$ and the fluid dynamic pressure p are constant at infinity.



Sketch 1

The position vector is denoted by $\mathbf{r} = \mathbf{i}_x x + \mathbf{i}_y y = \mathbf{i}_r r + \mathbf{i}_\theta \theta$, the magnetic induction vector by $\mathbf{B} = \mathbf{i}_x B_x + \mathbf{i}_y B_y$, the velocity vector by $\mathbf{q} = \mathbf{i}_x u + \mathbf{i}_y v$, the gradient operator by $\nabla = \mathbf{i}_x(\partial/\partial x) + \mathbf{i}_y(\partial/\partial y)$, the Laplacian operator by $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, the constant magnetic permeability by μ_0, and the constant density by ρ.

The flow is assumed to be described by the following differential equations and boundary conditions (the rationalized MKSQ system of units is used throughout):

$$\nabla \cdot \mathbf{q} = 0 \qquad \text{(continuity)} \tag{1}$$

$$(\mathbf{q} \cdot \nabla)\,\mathbf{q} + \frac{1}{\rho}\nabla p = \frac{1}{\rho\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} + \nu\nabla^2\mathbf{q} \qquad \text{(momentum)} \tag{2}$$

$$\nabla^2\mathbf{B} + \sigma\mu_0 \nabla \times (\mathbf{q} \times \mathbf{B}) = 0 \qquad \text{(induction)} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

$$p(\infty) = p_\infty, \quad \mathbf{q}(\infty) = \mathbf{i}_x U, \quad \mathbf{B}(\infty) = 0 \tag{5}$$

$$\lim_{r \to 0} \oint_{C_r} \mathbf{B} \cdot d\mathbf{r} = \mu_0 I \tag{6}$$

The path C_r is a circle of radius r about the origin; therefore, Eq. (6) is the mathematical interpretation of the current element.

III. NON-DIMENSIONALIZING

Three characteristic lengths appear in the problem, namely:

1. A length based on the current, $L_1 = (I/U)\sqrt{\mu_0/\rho}$
2. A magnetic diffusion length, $L_2 = 1/(\sigma\mu_0 U)$
3. A length based on viscous diffusion, $L_3 = \nu/U$

From these three lengths, it is seen that the problem has exactly two characteristic dimensionless parameters. In the following analysis it is assumed that one of these parameters, $\alpha = L_2/L_3$, is $O(1)$, and the other, $\epsilon = L_1/L_2$, is small. The plan of attack is to non-dimensionalize the differential equations, using L_2 as characteristic length and U as characteristic velocity. The parameters α and ϵ will appear in these non-dimensional equations, and the problem will be to find an asymptotic expansion of the solution, valid as $\epsilon \to 0$, by using a perturbation scheme.

The following dimensionless quantities are introduced:

$$\mathbf{q}^* = \frac{\mathbf{q}}{U} \qquad u^* = \frac{u}{U} \qquad v^* = \frac{v}{U} \tag{7}$$

$$p^* = \frac{p - p_\infty}{\rho U^2} \tag{8}$$

$$\mathbf{B}^* = \frac{\mathbf{B}}{\sigma I U \mu_0^2} \qquad B_x^* = \frac{B_x}{\sigma I U \mu_0^2} \qquad B_y^* = \frac{B_y}{\sigma I U \mu_0^2} \tag{9}$$

$$\mathbf{r}^* = \mathbf{r}\sigma\mu_0 U \qquad x^* = x\sigma\mu_0 U \qquad y^* = y\sigma\mu_0 U \tag{10}$$

Then the differential equations and boundary conditions become:

$$\nabla^* \cdot \mathbf{q}^* = 0 \tag{11}$$

$$(\mathbf{q}^* \cdot \nabla^*)\mathbf{q}^* + \nabla^* p^* = \epsilon^2(\nabla^* \times \mathbf{B}^*) \times \mathbf{B}^* + \alpha\nabla^{*2}\mathbf{q}^* \tag{12}$$

$$\nabla^{*2}\mathbf{B}^* + \nabla^* \times (\mathbf{q}^* \times \mathbf{B}^*) = 0 \tag{13}$$

$$\nabla^* \cdot \mathbf{B}^* = 0 \tag{14}$$

$$\mathbf{q}^*(\infty) = \mathbf{i}_x \qquad p^*(\infty) = 0 \qquad \mathbf{B}^*(\infty) = 0 \tag{15}$$

$$\lim_{r^* \to 0} \int_{C_r^*} \mathbf{B}^* \cdot d\mathbf{r}^* = 1 \tag{16}$$

where

$$\nabla^* = \mathbf{i}_x \frac{\partial}{\partial x^*} + \mathbf{i}_y \frac{\partial}{\partial y^*} \qquad \alpha = \nu\sigma\mu_0 \qquad \epsilon = \sigma l \sqrt{\frac{\mu_0^3}{\rho}}$$

and C_r^* is a circle of radius r^* about the origin.

IV. CONSTRUCTION OF THE ASYMPTOTIC EXPANSION

With the intent of finding an asymptotic expansion, assume a solution of the form

$$\mathbf{q}^*(\mathbf{r}^*;\epsilon) = \mathbf{q}^{(0)}(\mathbf{r}^*) + f_1(\epsilon)\,\mathbf{q}^{(1)}(\mathbf{r}^*) + \cdots \tag{17}$$

$$\mathbf{B}^*(\mathbf{r}^*;\epsilon) = \mathbf{B}^{(0)}(\mathbf{r}^*) + g_1(\epsilon)\,\mathbf{B}^{(1)}(\mathbf{r}^*) + \cdots \tag{18}$$

$$p^*(\mathbf{r}^*;\epsilon) = p^{(0)}(\mathbf{r}^*) + l_1(\epsilon)\,p^{(1)}(\mathbf{r}^*) + \cdots \tag{19}$$

When we substitute these equations into (11) and (12), and equate coefficients of ϵ, we obtain for the zeroth order

$$\nabla^* \cdot \mathbf{q}^{(0)} = 0 \tag{20}$$

$$(\mathbf{q}^{(0)} \cdot \nabla^*)\,\mathbf{q}^{(0)} + \nabla^* p^{(0)} = \alpha \nabla^{*2} \mathbf{q}^{(0)} \tag{21}$$

The solution of (20) and (21) is easily obtained by the following physical argument: Equations (20) and (21) are obtained formally from (11) and (12) by holding $\mathbf{r}^*$ fixed and letting $\epsilon \to 0$. That is, we are implicitly stating that, a priori, Eq. (20) and (21) describe the flow when $\epsilon \to 0$, at least in the region where $0 < |\mathbf{r}^*| = O(1)$, or, alternately, $0 < |\mathbf{r}| = O(1/\sigma\mu_0 U) = O(L_2)$. Now $|\mathbf{r}| = O(1/\sigma\mu_0 U)$ will remain constant as $\epsilon \to 0$ if we let $\epsilon \to 0$ by letting the current tend to zero while keeping the other parameters in $\epsilon = \sigma I\sqrt{\mu_0^3/\rho}$ fixed. Then physical reasoning indicates that the limiting velocity is simply the uniform stream velocity. That is,

$$\mathbf{q}^{(0)}(\mathbf{r}^*) = \mathbf{i}_x \tag{22}$$

Equation (21) is satisfied by (22) if it is assumed that $p^{(0)}(r^*) =$ constant, and from the boundary condition at infinity (Eq. 15) this constant must be zero. Thus

$$p^{(0)}(\mathbf{r}^*) = 0 \tag{23}$$

A similar argument provides a result which is used in Section VIII. This argument is as follows. In a region in which viscous effects are of secondary importance (for example, outside a viscous boundary layer) but at a distance $|\mathbf{r}|$, $0 < |\mathbf{r}| = O\,[(I/U)\sqrt{\mu_0/\rho}] = O(L_1)$ from the current element, the linearized velocity is still valid. (Note that

$L_1/L_2 = \epsilon$, so we are saying that the linearization is valid much closer to the current element than is indicated by the discussion leading to (22), so long as viscous effects can be ignored.) To see this, note that $\mathbf{r}$, $0 < |\mathbf{r}| = O(L_1)$, will remain constant as $\epsilon \to 0$ if the current in the current element is held fixed and ϵ is made small by letting $\sigma \to 0$ while the other parameters in $\epsilon = \sigma I \sqrt{\mu_0^3/\rho}$ are kept constant. This means, of course, that we are decoupling the velocity and magnetic fields, and thus the linearization of velocity is valid in the magnetohydrodynamic case whenever it is valid in the ordinary fluid-dynamic case.

To continue, let us substitute the assumed expansions (17), (18), (19) into Eq. (13) and (14) and equate coefficients of ϵ. The zeroth-order terms, using (22), give

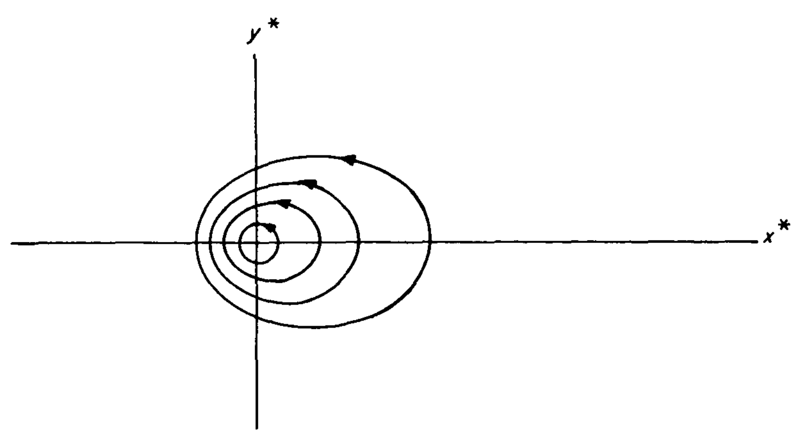
$$\nabla^{*2}\mathbf{B}^{(0)} - \frac{\partial \mathbf{B}^{(0)}}{\partial x^*} = 0 \tag{24}$$

$$\nabla^* \cdot \mathbf{B}^{(0)} = 0 \tag{25}$$

The solenoidal solution which satisfies the integral relation (16) can be verified to be

$$\mathbf{B}^{(0)}(\mathbf{r}^*) = \left\{ \mathbf{i}_x \left[- \frac{e^{\frac{x^*}{2}}\, y^*}{4\pi r^*} K_1\left(\frac{r^*}{2}\right)\right] + \mathbf{i}_y \left[\frac{e^{\frac{x^*}{2}}\, x^*}{4\pi r^*} K_1\left(\frac{r^*}{2}\right) - \frac{e^{\frac{x^*}{2}}}{4\pi} K_0\left(\frac{r^*}{2}\right)\right]\right\} \tag{26}$$

where $K_0\,(r^*/2)$, $K_1(r^*/2)$ are the modified Bessel functions of the second kind of the zeroth and first orders, with argument $r^*/2$. Sketch 2 shows the lines of magnetic induction.



Sketch 2

The perturbation on the velocity and pressure fields are obtained next. If the assumed expansions (17), (18), and (19) are substituted into Eq. (11) and (12), and the largest unknown terms are retained, we obtain

$$f_1(\epsilon)\,\nabla^* \cdot \mathbf{q}^{(1)} = 0 \tag{27}$$

$$f_1(\epsilon)\,\alpha\nabla^{*2}\mathbf{q}^{(1)} - f_1(\epsilon)\frac{\partial \mathbf{q}^{(1)}}{\partial x^*} - l_1(\epsilon)\,\nabla^* p^{(1)} = -\epsilon^2\,(\nabla^* \times \mathbf{B}^{(0)}) \times \mathbf{B}^{(0)} \tag{28}$$

Hence

$$f_1(\epsilon) = l_1(\epsilon) = \epsilon^2 \tag{29}$$

Written in full, using Eq. (26) for $\mathbf{B}^{(0)}$, we have

$$\nabla^* \cdot \mathbf{q}^{(1)} = 0 \tag{30}$$

$$\begin{aligned}\alpha\nabla^{*2}\mathbf{q}^{(1)} - \frac{\partial \mathbf{q}^{(1)}}{\partial x^*} - \nabla^* p^{(1)} = \mathbf{i}_x\Bigg[&\frac{e^{x^*}}{16\pi^2}\,\frac{x^{*2}}{r^{*2}}\,K_1\left(\frac{r^*}{2}\right)K_1\left(\frac{r^*}{2}\right) \\ &- \frac{e^{x^*}x^*}{8\pi^2 r^*}\,K_0\left(\frac{r^*}{2}\right)K_1\left(\frac{r^*}{2}\right) + \frac{e^{x^*}}{16\pi^2}\,K_0\left(\frac{r^*}{2}\right)K_0\left(\frac{r^*}{2}\right)\Bigg] \\ + \mathbf{i}_y\Bigg[&-\frac{e^{x^*}y^*}{16\pi^2 r^*}\,K_0\left(\frac{r^*}{2}\right)K_1\left(\frac{r^*}{2}\right) + \frac{e^{x^*}x^*y^*}{16\pi^2 r^{*2}}\,K_1\left(\frac{r^*}{2}\right)K_1\left(\frac{r^*}{2}\right)\Bigg]\end{aligned} \tag{31}$$

If the right side of (31) were zero, (30) and (31) would be Oseen's equations. We can obtain a formal particular solution of Eq. (30) and (31) in the form of an integral by using the fundamental solution tensor of the Oseen equations (Ref. 2). For example, the expression for $\epsilon^2 u_p^{(1)}(\mathbf{r}^*)$ (the subscript p indicates that this is a particular solution) is, formally,

$$\epsilon^2 u_p^{(1)}(\mathbf{r}^*) = \frac{\epsilon^2}{2\pi} \int_0^\infty \int_0^{2\pi} \left\{ F(\mathbf{r}^*; \rho^*, \phi) \left[-\frac{e^{\xi^*}}{16\pi^2} \frac{\xi^{*2}}{\rho^{*2}} K_1\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right) \right. \right.$$

$$\left. + \frac{e^{\xi^*}\xi^*}{8\pi^2 \rho^*} K_0\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right) - \frac{e^{\xi^*}}{16\pi^2} K_0\left(\frac{\rho^*}{2}\right) K_0\left(\frac{\rho^*}{2}\right) \right]$$

$$+ \left[-\frac{y^* - \eta^*}{|\mathbf{r}^* - \boldsymbol{\rho}^*|^2} + \frac{1}{2\alpha} K_1\left(\frac{|\mathbf{r}^* - \boldsymbol{\rho}^*|}{2\alpha}\right) \frac{y^* - \eta^*}{|\mathbf{r}^* - \boldsymbol{\rho}^*|} e^{\frac{x^* - \xi^*}{2\alpha}} \right]$$

$$\left. \times \left[\frac{e^{\xi^*}\eta^*}{16\pi^2 \rho^*} K_0\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right) - \frac{e^{\xi^*}\xi^*\eta^*}{16\pi^2 \rho^{*2}} K_1\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right) \right] \right\} \rho^* \, d\phi \, d\rho^* \tag{32}$$

where

$$F(\mathbf{r}^*; \rho^*, \phi) = \frac{1}{2\alpha} K_0\left(\frac{|\mathbf{r}^* - \boldsymbol{\rho}^*|}{2\alpha}\right) e^{\frac{x^* - \xi^*}{2\alpha}} - \frac{x^* - \xi^*}{|\mathbf{r}^* - \boldsymbol{\rho}^*|^2} + \frac{1}{2\alpha} K_1\left(\frac{|\mathbf{r}^* - \boldsymbol{\rho}^*|}{2\alpha}\right) \frac{x^* - \xi^*}{|\mathbf{r}^* - \boldsymbol{\rho}^*|} e^{\frac{x^* - \xi^*}{2\alpha}} \tag{33}$$

and where the variable of integration is $\boldsymbol{\rho}^* = \mathbf{i}_x \xi^* + \mathbf{i}_y \eta^* = \mathbf{i}_\rho \rho^* + \mathbf{i}_\phi \phi$ in Cartesian and polar coordinates, respectively. The above integral does not exist in the usual sense, but, by taking the "finite part" (Ref. 3, page 38), we obtain a solution of the differential equations. Using the standard methods for expressing the finite part, we have

$$\epsilon^2 u_p^{(1)}(\mathbf{r}^*) = \frac{\epsilon^2}{2\pi}\int_0^1\int_0^{2\pi}\frac{1}{\rho^*}\left\{F(\mathbf{r}^*;\rho^*;\phi)\left[-\frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2}\rho^{*2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right]\right.$$

$$\left.-4F(\mathbf{r}^*;0,\phi)\left(-\frac{\cos^2\phi}{16\pi^2}\right)\right\}d\phi\, d\rho^*$$

$$+\frac{\epsilon^2}{2\pi}\int_1^{\infty}\int_0^{2\pi}\frac{1}{\rho^*}\left\{F(\mathbf{r}^*;\rho^*,\phi)\left[-\frac{e^{\rho^*\cos\phi}\cos^2\phi}{16\pi^2}\rho^{*2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right]\right\}d\phi\, d\rho^*$$

$$+\frac{\epsilon^2}{2\pi}\int_0^{\infty}\int_0^{2\pi}F(\mathbf{r}^*;\rho^*,\phi)\left[\frac{e^{\rho^*\cos\phi}\cos\phi}{8\pi^2}K_0\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)-\frac{e^{\rho^*\cos\phi}}{16\pi^2}K_0\left(\frac{\rho^*}{2}\right)K_0\left(\frac{\rho^*}{2}\right)\right]$$

$$\times\ \rho^*\, d\phi\, d\rho^*$$

$$+\frac{\epsilon^2}{2\pi}\int_0^{\infty}\int_0^{2\pi}\left[-\frac{y^*-\rho^*\sin\phi}{|\mathbf{r}^*-\boldsymbol{\rho}^*|^2}+\frac{1}{2\alpha}K_1\left(\frac{|\mathbf{r}^*-\boldsymbol{\rho}^*|}{2\alpha}\right)\frac{y^*-\rho^*\sin\phi}{|\mathbf{r}^*-\boldsymbol{\rho}^*|}e^{\frac{x^*-\rho^*\cos\phi}{2\alpha}}\right]$$

$$\times\left[\frac{e^{\rho^*\cos\phi}\sin\phi}{16\pi^2}K_0\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)-\frac{e^{\rho^*\cos\phi}\cos\phi\sin\phi}{16\pi^2}K_1\left(\frac{\rho^*}{2}\right)K_1\left(\frac{\rho^*}{2}\right)\right]\rho^*\,d\phi\, d\rho^*$$

(34)

where $F(\mathbf{r}^*;\rho^*,\phi)$ is given by (33). The above discussion, with obvious modifications, applies also to $\epsilon^2 p_p^{(1)}(\mathbf{r}^*)$ and $\epsilon^2 v_p^{(1)}(\mathbf{r}^*)$.

A somewhat more physically intuitive way of arriving at (34) is as follows. We note that the integral (32) has a logarithmic infinity at the origin. If we temporarily assume that the forcing terms are zero for $|\mathbf{r}^*| \leq R > 0$, then the lower limit of integration with respect to ρ^* is R, and the integral will converge. Suppose we now add to this resulting integral the following term, proportional to the well-known fundamental solution of Oseen's equations (Ref. 2):

$$- \frac{1}{8\pi^2} \log R \left[\frac{1}{2\alpha} K_0 \left(\frac{r^*}{2} \right) e^{\frac{x^*}{2\alpha}} - \frac{x^*}{r^{*2}} + \frac{1}{2\alpha} K_1 \left(\frac{r^*}{2\alpha} \right) \frac{x^*}{r^{*2}} e^{\frac{x^*}{2\alpha}} \right] = - \frac{1}{8\pi^2} \log R \, F(\mathbf{r}^*; 0, \phi) \tag{35}$$

$$= \frac{1}{2\pi} \int_R^1 \int_0^{2\pi} 4 F(\mathbf{r}^*; 0, \phi) \frac{\cos^2 \phi}{16\pi^2 \rho^*} \, d\phi \, d\rho^* \tag{36}$$

where $F(\mathbf{r}^*; \rho^*, \phi)$ is given by Eq. (33). The sum also satisfies the differential equations with the modified non-homogeneous terms. Now, let $R \to 0$. The result is identical to (34).

To the particular solution (34) can be added a term proportional to the fundamental solution of the Oseen equations. For the current element case, it is difficult to determine exactly what constant of proportionality to use, and it is not discussed further in this paper.

V. DRAG DUE TO $\mathbf{B}^{(0)}$, $q_p^{(1)}$, $p_p^{(1)}$

Without entering into a detailed discussion, we can make a few comments about the consequences of the expressions for the solution obtained in Section IV. First of all, it is easily verified that when $\mathbf{B}^{(0)}$, $\mathbf{q}_p^{(1)}$, $p_p^{(1)}$ are written for small values of $\mathbf{r}^*$, there appear viscous stress terms, pressure terms, and Maxwell stress terms, each with a singular behavior of the form

$$\frac{\log r^*}{r^*} \tag{37}$$

Fortunately, these terms all exactly cancel, leaving no infinite net forces. In addition, there are terms with a singular behavior of the form

$$\frac{1}{r^*} \tag{38}$$

which do not cancel, and thus there is a finite drag on the current element. It is this drag with which we shall now concern ourselves.

As is well known, we can find the drag by considering the flux of momentum arising from the velocity, pressure, and magnetic induction fields at infinity. The magnetic stress terms from $\mathbf{B}^{(0)}$ clearly behave, for large r, like

$$\frac{e^{-r^*+x^*}}{r^*} \tag{39}$$

which is a sufficiently rapid decay to give zero contribution to momentum flux, so we need contend with only $\mathbf{q}_p^{(1)}(\mathbf{r}^*)$ and $p_p^{(1)}(\mathbf{r}^*)$. The actual computation can be reduced to a single integration which must be carried out numerically. An intuitive argument leading to this integral is as follows (for a proof, see Ref. 1, appendix). We know from the symmetry of the problem that the force on the current element is in the x-direction; i.e., there is no lift. As we move away from the origin, the forcing terms, if they decay sufficiently fast at infinity, gradually appear as though concentrated at the origin and directed along the x-axis, so that for large $\mathbf{r}^*$, $\mathbf{q}_p^{(1)}$ and $p_p^{(1)}$ behave like the response to a concentrated force at the origin, i.e., proportional to the fundamental solution; and the constant of proportionality is given by the integral of the x-direction forcing terms in the integrand. This constant corresponds to the product of $\sigma\mu_0$ and the "source strength," expressed in physical variables, of the so-called longitudinal (i.e., irrotational) component, and it is well-known that the drag in physical variables is the product of ρU and this source strength. If the integral is negative, the contribution to drag is positive.

The integral of the x-direction forcing terms can be conveniently obtained from Eq. (34). We need only those terms even in y, giving the result

$$J = \epsilon^2 \int_0^1 \int_0^{2\pi} \frac{1}{\rho^*}\left[- \frac{e^{\rho^* \cos\phi} \cos^2\phi}{16\pi^2} \rho^{*2} K_1\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right) + \frac{\cos^2\phi}{4\pi^2}\right] d\phi\, d\rho^*$$

$$+ \epsilon^2 \int_1^\infty \int_0^{2\pi} \left[- \frac{e^{\rho^* \cos\phi} \cos^2\phi}{16\pi^2} K_1\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right)\right] \rho^* d\phi\, d\rho^*$$

$$+ \epsilon^2 \int_0^\infty \int_0^{2\pi} \left[\frac{e^{\rho^* \cos\phi} \cos\phi}{8\pi^2} K_0\left(\frac{\rho^*}{2}\right) K_1\left(\frac{\rho^*}{2}\right) - \frac{e^{\rho^* \cos\phi}}{16\pi^2} K_0\left(\frac{\rho^*}{2}\right) K_0\left(\frac{\rho^*}{2}\right)\right] \rho^* d\phi\, d\rho^* \tag{40}$$

The result of the numerical integration is

$$J \cong -\epsilon^2 \frac{1.90}{16\pi^2} \tag{41}$$

As stated above, the drag is the product of J, $1/\sigma\mu_0$, and ρU. Thus

$$Drag_p \cong \frac{1.9}{16\pi^2} \sigma I^2 \mu_0^2 U \text{ newtons/meter} \tag{42}$$

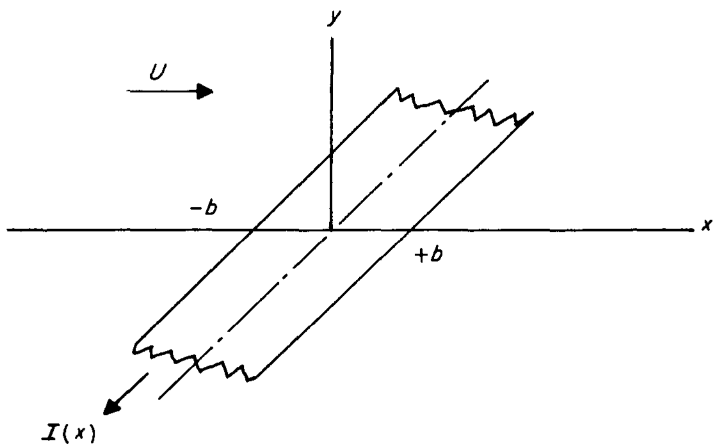
The subscript p indicates that this is the drag contribution from the particular solutions $\mathbf{q}_p^{(1)}$, $\mathbf{p}_p^{(1)}$ and $\mathbf{B}^{(0)}$

Section VI deals with an application of the preceding results.

VI. FLOW PAST A CURRENT-CARRYING FLAT PLATE

In the preceding sections we have considered the flow past a current element. Results obtained there are now applied to the study of the flow past a finite flat plate. Some remarks as to the validity of the technique will be made in Section VIII.

Consider a flat plate with leading edge at $(-b, 0)$ and trailing edge at $(b, 0)$, as shown in Sketch 3.



Sketch 3

The plate carries a current, I_0 amperes, and as a specific example we assume that the current density in the plate is uniform,

$$I(x) = \frac{I_0}{2b} \text{ amperes/meter} \qquad -b \leq x \leq b \tag{43}$$

Now assume that Eq. (34) has been evaluated for values of $\mathbf{r}^*$ on the x^* axis. If the current element is on the x^* axis at ξ^*, and the current element carries current dI, then the non-dimensional perturbation velocity at x^* will be

$$\epsilon^2 u_p^{(1)} \left[\mathbf{i}_x (x^* - \xi^*) \right] \frac{dI}{I_0} \tag{44}$$

and the effect induced by the entire flat plate will thus be simply

$$u_A(x^*) = \int_{-b^*}^{b^*} \epsilon^2 u_p^{(1)} [\mathrm{i}_x(x^* - \xi^*)] \frac{I^*(\xi^*)}{I_0} d\xi^* \tag{45}$$

$$= \frac{\epsilon^2}{2b^*} \int_{-b^*}^{b^*} u_p^{(1)} [\mathrm{i}_x(x^* - \xi^*)] d\xi^* \tag{46}$$

where

$$I^*(\xi^*) = \frac{I_0}{2b^*}$$

To this solution we can add a homogeneous solution, and the choice is based on the requirement that the no-slip condition on the plate be satisfied. The x-direction non-dimensional velocity perturbation at x^* due to a singular force of strength $\rho U/\sigma\mu_0$ (i.e., a unit dimensionless "force") placed on the x-axis at the point ξ^* and acting in the upstream direction is denoted by $u_0 [\mathrm{i}_x(x^* - \xi^*)]$; $u_0 [\mathrm{i}_x(x^* - \xi^*)]$ is then simply the fundamental solution of Eq. (30) and (31), the right-hand side of (31) being replaced by a Dirac delta function. If we have a distribution of such forces $f(\xi^*)$ along the plate, they induce at the point x^* a velocity (see Ref. 4, p. 107)

$$u_H(x^*) = \int_{-b^*}^{b^*} u_0 [\mathrm{i}_x(x^* - \xi^*)] f(\xi^*) d\xi^* \tag{47}$$

To satisfy the no-slip condition, we thus require

$$1 + u_H(x^*) + u_A(x^*) = 0 \qquad -b^* \leq x^* \leq b^* \tag{48}$$

the first term being the dimensionless free-stream velocity. Using Eq. (47), Eq. (48) becomes

$$\int_{-b^*}^{b^*} u_0 [\mathrm{i}_x(x^* - \xi^*)] f(\xi^*) d\xi^* = -1 - u_A(x^*) \tag{49}$$

which is a Fredholm integral equation of the first kind with the unknown function $f(\xi^*)$. It is convenient to write

$$f(\xi^*) = f_v(\xi^*) + f_A(\xi^*) \tag{50}$$

where

$$\int_{-b^*}^{b^*} u_0 \left[\mathbf{i}_x (x^* - \xi^*)\right] f_v(\xi^*)\, d\xi^* = -1 \tag{51}$$

and

$$\int_{-b^*}^{b^*} u_0 \left[\mathbf{i}_x (x^* - \xi^*)\right] f_A(\xi^*)\, d\xi^* = -u_A(x^*) \tag{52}$$

Equation (51) has been discussed by Bairstow, Cave, and Lang (Ref. 5) and others. The contribution to drag from $u_H(x^*)$ is simply

$$Drag_H = \frac{\rho U}{\sigma \mu_0} \int_{-b^*}^{b^*} \left[f_v(\xi^*) + f_A(\xi^*)\right] d\xi^* = Drag_v + Drag_A \tag{53}$$

where $Drag_v$ is the fluid dynamic drag when no currents flow. The total drag is the sum of the contributions from (42) and (53),

$$Drag_{total} = Drag_v + Drag_A + Drag_p \tag{54}$$

It can be shown that as the length of the plate becomes very small, the u_A behaves like $(\log b)^2/(\alpha\, 32\pi^2)$. This means that the velocity past the plate tends to be greater than the free-stream velocity; consequently the skin friction is greater than that for the non-magnetohydrodynamic case. However, as the length of the plate increases, this effect tends to be reversed, because the velocity induced by those current elements some distance away from a given point on the plate is in a direction opposing the free stream. A more complete discussion must be based on evaluation of Eq. (46), which in turn depends on the evaluation of u_p.

VII. EXPERIMENTAL POSSIBILITIES

Some indications as to the possibility of experimentally verifying the theory of this paper in the laboratory can be given by considering an experiment in which liquid sodium flows at 1 meter per sec past a thin plate 1 cm wide (in the direction of the flow), carrying a current of 2000 amperes. The quantity ϵ^2, which we are assuming to be small, is in this case $\approx 1/2$. (A sodium magnetohydrodynamic experimental facility is in the early stages of development at the Laboratory.) For sodium, typical physical parameters are

viscosity $\mu = 5.3 \times 10^{-3}$ poise

density $\rho = 9.2 \times 10^2$ kgm/meter^3

conductivity $\sigma = 8.55 \times 10^6$ mho/meter

permeability of free space, $\mu_0 = 4\pi \times 10^{-7}$ henry/meter

From Eq. (42), we have a contribution to drag of

$$Drag_p \cong \frac{1.9}{16\pi^2} \sigma I^2 \mu_0^2 U \cong 0.65 \text{ newtons/meter} \tag{55}$$

The fluid dynamic drag on the flat plate (i.e., when there is no magnetohydrodynamic effect) is the product of ρU^2, the length of the plate, and the drag coefficient, C_D, which is near unity for this case (Reynolds number is several thousand); therefore

$$\begin{aligned} Drag_{fluid\ dynamic} &\cong \rho U^2 L \cdot C_D \\ &\cong 9 \text{ newtons/meter} \end{aligned} \tag{56}$$

Of course, this does not take into account the other contribution to the magnetohydrodynamic drag, namely $Drag_A$, from Eq. (53). As mentioned above, this contribution must await computation of u_p. However, comparison of $Drag_{fluid\ dynamic}$ and $Drag_p$ does indicate a definite possibility of measuring the magnetohydrodynamic effect. In any case, note that since the induced dimensionless velocity at the plate u_A is proportional to ϵ^2, Eq. (52) shows $f_A(\xi^*)$ is also proportional to ϵ^2, so that, in physical variables, $Drag_A$ is proportional to

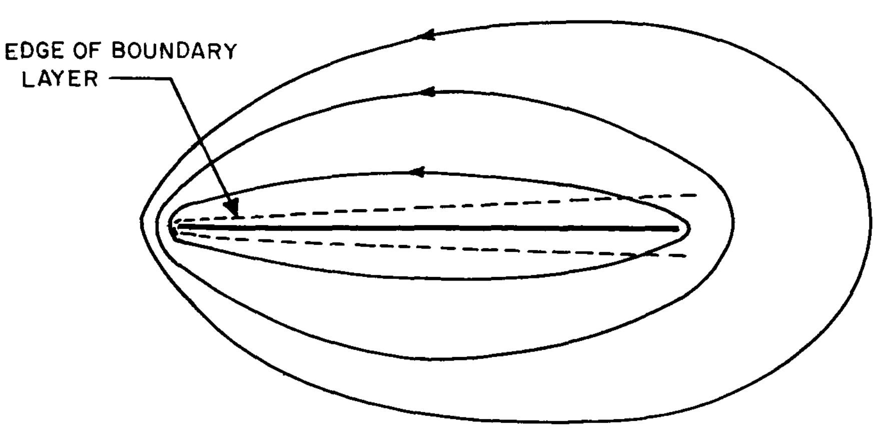
$$\frac{\rho U}{\sigma \mu_0} \epsilon^2 = \sigma I^2 \mu_0^2 U$$

Since $Drag_p$ is proportional to this same coefficient, we see immediately that the total magnetohydrodynamic contribution to drag varies as the square of the current. It seems that this behavior might be verified experimentally, even when we do not know the numerical value of $Drag_A$ and hence of $Drag_A + Drag_p$.

Experimentally the situation is pleasant because the fluid dynamic drag is proportional to the length of the plate, and, roughly speaking, the total current that can be used in the plate is proportional to the length of plate. But the magnetohydrodynamic drag contribution, being proportional to the square of the current, varies as the square of the length of the plate. This fact may be used to advantage in an experimental arrangement. (The reader should be cautioned that this behavior applies to $Drag_p$ only.)

VIII. VALIDITY OF THE APPROXIMATION

The validity of the analysis may be discussed by considering the regions in which the linearization is valid. For the ordinary fluid dynamic case, many workers, including Bairstow, Cave, and Lang (Ref. 5), Piercy and Winny (Ref. 6), and Lewis and Carrier (Ref. 7), have shown that, quantitatively, the use of Oseen's equations to find the drag due to flow past a flat plate is not very accurate (although qualitatively the equations provide a nice picture of the flow). The explanation for the failure can be traced to the fact that the skin friction coefficient must be found in that very region where the linearization breaks down. In the magnetohydrodynamic case, this argument indicates that we should not expect that the contribution to drag given by (50) will be any more accurate than the solution based on Oseen's equations in the ordinary fluid dynamic case. On the other hand, under a suitable condition the contribution from (42) should be accurate. The required "suitable condition" is that the boundary layer thickness be small compared to L_1, the current length. Roughly speaking, this is because the approximation used for the body forces in the momentum equation (28) is valid up to the point where the linearization breaks down. From the remarks made in Section IV we know that the magnetic field does not spoil the linearization outside the boundary layer. Hence it seems probable that the linearization is valid everywhere outside the boundary layer, even in the magnetohydrodynamic case. Thus the expression for the body forces is valid up to the edge of the boundary layer, and if this layer is thin the assumption that these forces extend right up to the plate (as was assumed in the analysis) will be very nearly valid (see Sketch 4).



Sketch 4

For the liquid sodium experiment mentioned above, the boundary layer thickness attains a maximum at the trailing edge of

$$\delta \cong \sqrt{\frac{\nu L}{U}}$$

$$\cong 0.024 \text{ cm} \tag{57}$$

whereas L_1 in this case is

$$L_1 = \frac{I_0}{U}\sqrt{\frac{\mu_0}{\rho}}$$

$$\cong 7.5 \text{ cm} \tag{58}$$

Hence $L_1 \gg \delta$, so an experiment using liquid sodium should provide a valid test for the theory.

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